Problem Set 5 – Relativity (G0I36A)

1 Hands-on with Christoffel Symbols and the Riemann Tensor

1.) First, note that $g_{rr} = 1$, $g_{\theta\theta} = r^2$, and $g_{\phi\phi} = r^2 \sin^2\theta$. The corresponding inverse metric components are $g^{rr} = 1$, $g^{\theta\theta} = \frac{1}{r^2}$, and $g^{\phi\phi} = \frac{1}{r^2 \sin^2\theta}$, such that $g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu$.

Notice that $g_{\mu\nu}$ is diagonal, so $g_{\mu\nu} = 0$ unless $\mu = \nu$. Therefore the only way to get a non-trivial Christoffel symbol $\Gamma^\rho_{\mu\nu}$ is to have one of the indices appear at least twice. However, we cannot have any index repeated three times; g_{rr} is indepenent of r, $g_{\theta\theta}$ is independent of θ, and $g_{\phi\phi}$ is independent of ϕ, and so $\partial_\mu g_{\nu\rho} = 0$ when all indices are equal. So, all of the non-trivial Christoffel symbols will have one index repeated exactly twice.

Additionally, all components of the metric depend on r or θ, not ϕ. So, symbols like $\Gamma^\phi_{rr} = \frac{1}{2} g^{\phi\phi} (-\partial_\phi g_{rr})$ and $\Gamma^\phi_{\theta\theta} = \frac{1}{2} g^{\phi\phi} (-\partial_\phi g_{\theta\theta})$ will vanish automatically. Similarly, so will $\Gamma^r_{\phi r}$ and $\Gamma^\theta_{\phi\theta}$. And, since g_{rr} is constant and doesn't depend on θ, $\Gamma^r_{\theta r}$ and Γ^θ_{rr} will also vanish by similar reasoning.

Lastly, the Christoffel symbols are symmetric in their lower indices, and so $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. Taking this all into account, the non-trivial independent Christoffel symbols are the following:

$$
\begin{aligned}
&\Gamma^r_{\theta\theta} \ , \quad \Gamma^r_{\phi\phi} \ , \\
&\Gamma^\theta_{r\theta} \ , \quad \Gamma^\theta_{\phi\phi} \ , \\
&\Gamma^\phi_{r\phi} \ , \quad \Gamma^\phi_{\theta\phi} \ .
\end{aligned} \tag{1.1}
$$

All others vanish, or can be obtained by symmetry (e.g. $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta}$).

Explicitly, their computation is as follows:

$$
\begin{aligned}
\Gamma^r_{\theta\theta} &= \frac{1}{2} g^{rr} (-\partial_r g_{\theta\theta}) = -r \ , \\
\Gamma^r_{\phi\phi} &= \frac{1}{2} g^{rr} (-\partial_r g_{\phi\phi}) = -r \sin^2\theta \ , \\
\Gamma^\theta_{r\theta} &= \frac{1}{2} g^{\theta\theta} (\partial_r g_{\theta\theta}) = \frac{1}{r} \ , \\
\Gamma^\theta_{\phi\phi} &= \frac{1}{2} g^{\theta\theta} (-\partial_\theta g_{\phi\phi}) = -\sin\theta\cos\theta \ , \\
\Gamma^\phi_{r\phi} &= \frac{1}{2} g^{\phi\phi} (\partial_r g_{\phi\phi}) = \frac{1}{r} \ , \\
\Gamma^\phi_{\theta\phi} &= \frac{1}{2} g^{\phi\phi} (\partial_\theta g_{\phi\phi}) = \cot\theta \ .
\end{aligned} \tag{1.2}
$$

Note that we heavily used the fact that the metric is diagonal when writing down these expressions. In particular, it is very important that $g^{r\sigma}\partial_\sigma = g^{rr}\partial_r$, $g^{\theta\sigma}\partial_\sigma = \partial_\theta$, etc.

2.) The Riemann tensor is antisymmetric in its first and second pairs of indices, but symmetric under exchange of these two pairs:

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho} = R_{\rho\sigma\mu\nu} \ . \tag{1.3}$$

Based on the antisymmetry, the first and second pairs cannot have the same index appearing twice. But, since we are in 3D and the Riemann tensor has four components, at least one of the indices must appear at least twice. Combining these facts tells us that no index can appear three times or four times; thinks like $R_{rrr\theta} = 0$ by antisymmetry.

Additionally, symmetry of swapping pairs greatly reduces the number of independent Riemann tensor components. Things like e.g. $R_{r\phi r\theta}$ are equal to $R_{r\theta r\phi}$.

After considering this all, we are left with the following non-trivial symbols:

$$\begin{aligned} &R_{r\theta r\theta} \ , \quad R_{r\phi r\phi} \ , \quad R_{\theta\phi\theta\phi} \ , \\ &R_{r\theta r\phi} \ , \quad R_{\theta r\theta\phi} \ , \quad R_{\phi r\phi\theta} \ . \end{aligned} \tag{1.4}$$

All of these are such that one index must appear exactly twice and the others once, none of them vanish by antisymmetry, and all other non-zero Riemann tensor components are obtained by swapping around indices from this set of six.

Note that it's easier to compute $R^{\mu}{}_{\nu\rho\sigma}$ than $R_{\mu\nu\rho\sigma}$, given the way we usually write the formula for the Riemann tensor. But, since the metric is diagonal, this is no problem. We can easily see that, for example, $R_{r\theta r\theta} = g_{rr} R^{r}{}_{\theta r\theta}$, i.e. we can easily raise and lower indices just by multiplication. So, the six independent Riemann tensor components we will compute are

$$\begin{aligned} &R^{r}{}_{\theta r\theta} \ , \quad R^{r}{}_{\phi r\phi} \ , \quad R^{\theta}{}_{\phi\theta\phi} \ , \\ &R^{r}{}_{\theta r\phi} \ , \quad R^{\theta}{}_{r\theta\phi} \ , \quad R^{\phi}{}_{r\phi\theta} \ . \end{aligned} \tag{1.5}$$

We will compute the first explicitly:

$$\begin{aligned} R^{r}{}_{\theta r\theta} &= \partial_r \Gamma^{r}_{\theta\theta} - \partial_\theta \Gamma^{r}_{\theta r} + \Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{\theta\theta} - \Gamma^{r}_{\theta\lambda}\Gamma^{\lambda}_{\theta r} \\ &= \partial_r \left(-r\right) - \partial_\theta \left(0\right) + 0 - \Gamma^{r}_{\theta\theta}\Gamma^{\theta}_{\theta r} \\ &= -1 - (-r)(1/r) \\ &= 0 \ . \end{aligned} \tag{1.6}$$

Note that the index λ is summed over. All of the values it takes yields zero in the third term, while only the $\lambda = \theta$ component in the fourth term yields non-zero Christoffel symbols.

The next is similar:

$$\begin{aligned} R^{r}{}_{\phi r\phi} &= \partial_r \Gamma^{r}_{\phi\phi} - \partial_\phi \Gamma^{r}_{\phi r} + \Gamma^{r}_{r\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{r}_{\phi\lambda}\Gamma^{\lambda}_{\phi r} \\ &= \partial_r \left(-r\sin^2\theta\right) - \partial_\phi \left(0\right) + 0 - \Gamma^{r}_{\phi\phi}\Gamma^{\phi}_{\phi r} \\ &= -\sin^2\theta - \left(-r\sin^2\theta\right)(1/r) \\ &= -\sin^2\theta + \sin^2\theta \\ &= 0 \ . \end{aligned} \tag{1.7}$$

Once again, we get zero.

The third Riemann tensor component is probably the most non-trivial computation:

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\phi\theta} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\phi\theta} \\ &= \partial_\theta \left(-\sin\theta\,\cos\theta\right) - \partial_\phi\left(0\right) + \Gamma^\theta_{\theta r}\Gamma^r_{\phi\phi} - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} \\ &= -\cos^2\theta + \sin^2\theta + (1/r)\left(-r\sin^2\theta\right) - \left(-\sin\theta\,\cos\theta\right)\left(\cot\theta\right) \\ &= -\cos^2\theta + \sin^2\theta - \sin^2\theta + \cos^2\theta \\ &= 0\ . \end{aligned} \tag{1.8}$$

Once again, zero.

I leave showing that the remaining three independent components vanish up to you, dear reader. It's algorithmic at this point; write down the expression, plug in for the Christoffel symbols, try to see what is zero, and then show that the remaining non-zero terms all end up cancelling.

3.) All components of the Riemann tensor vanish, and so we are left simply with $R_{\mu\nu\rho\sigma} = 0$. Additionally, $R_{\mu\nu\rho\sigma}$ is related to the curvature of the spacetime, and any spacetime with $R_{\mu\nu\rho\sigma} = 0$ everywhere is just a flat spacetime. So, we have shown that our spacetime is flat.

This is because our metric is secretly just the Euclidean flat space metric $ds^2 = dx^2 + dy^2 + dz^2$ in disguise! Remember, we've shown explicitly in previous problem sets that there is a coordinate transformation that takes us from Cartesian to spherical coordinates, and so the associated spacetimes are the same.

4.) To solve this problem, first note that the (rr), $(\theta\theta)$, and $(\phi\phi)$ components of the metric and the inverse metric are the same as they were in the previous problems. Additionally, the metric is still diagonal, and g_{tt} is a constant. This means that *absolutely nothing changes in our computation of the Christoffel symbols!*

Nothing depends on t, so no derivatives with respect to t will be important. g_{tt} is constant, so all derivatives of it vanish. Therefore any Christoffel symbol with a t for one of its indices will vanish. And, since the old part of the metric is intact, the computation of the Christoffel symbols in problem 1 will still be exactly the same. Thus we are left with the same set of six independent non-zero Christoffel symbols, and we can use them in exactly the same way as before to show that the Riemann tensor components are all zero again, as we expect for a flat spacetime.

2 Curvature on the Two-Sphere

5.) We've done this a few times by now on previous problem sets. You just fix $r = L$, set $dr = 0$ in the metric (since a fixed coordinate has no infinitesimal variation), and you are left with

$$ds^2 = L^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\ . \tag{2.1}$$

So, $g_{\theta\theta} = L^2$, and $g_{\phi\phi} = L^2\sin^2\theta$. We then immediately also find $g^{\theta\theta} = \frac{1}{L^2}$ and $g^{\phi\phi} = \frac{1}{L^2\sin^2\theta}$.

6.) Since $g_{\theta\theta}$ is a constant, any derivatives of it vanish. This means that the only non-zero Christoffel symbols are ones where we take derivatives of $g_{\phi\phi}$. And, these derivatives must be θ-derivatives, since $g_{\phi\phi}$ is independent of ϕ. Therefore the only non-zero symbols have ϕ appearing twice and θ appearing once. The independent non-zero symbols are thus

$$\Gamma^\theta_{\phi\phi}\ , \quad \Gamma^\phi_{\theta\phi}\ , \tag{2.2}$$

since the third non-zero symbol $\Gamma^{\phi}_{\phi\theta}$ is equal to $\Gamma^{\phi}_{\theta\phi}$ by symmetry. We compute them as follows:

$$\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2} g^{\theta\theta} \left(-\partial_\theta g_{\phi\phi} \right) = -\sin\theta\cos\theta \ , \\
\Gamma^{\phi}_{\theta\phi} &= \frac{1}{2} g^{\phi\phi} \left(\partial_\theta g_{\phi\phi} \right) = \cot\theta \ .
\end{aligned} \tag{2.3}$$

These are the same as in 3D spherical coordinates, which makes sense; we have not altered the structure θ, ϕ pieces in the metric.

Now, for the Riemann tensor. Symmetries tell us that the only independent, non-zero component is simply $R_{\theta\phi\theta\phi}$. All other components (like $R_{\theta\theta\theta\phi} = -R_{\theta\theta\theta\phi} = 0$) vanish by antisymmetry, or can be obtained via symmetries (like $R_{\phi\theta\theta\phi} = -R_{\theta\phi\theta\phi}$) from this one component. The metric is diagonal, and so it also suffices just to compute $R^{\theta}{}_{\phi\theta\phi}$, since raising and lowering indices amounts to multiplication for diagonal metrics. This one independent component is computed as follows:

$$\begin{aligned}
R^{\theta}{}_{\phi\theta\phi} &= \partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\theta} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\phi\theta} \\
&= \partial_\theta \left(-\sin\theta\cos\theta \right) - \partial_\phi(0) + 0 - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta} \\
&= -\cos^2\theta + \sin^2\theta - (-\sin\theta\cos\theta)(\cot\theta) \\
&= -\cos^2\theta + \sin^2\theta + \cos^2\theta \\
&= \sin^2\theta \ .
\end{aligned} \tag{2.4}$$

This is *not* zero, unlike in the full 3D spherical coordinates. It was only when we had the r coordinate that we had additional Christoffel symbols that cancelled this $\sin^2\theta$ to make the Riemann tensor vanish.

The Ricci tensor is $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$. It is symmetric, and so there are three independent components $R_{\theta\theta}$, $R_{\phi\phi}$, and $R_{\theta\phi} = R_{\phi\theta}$. Computing them is straightforward by raising and lowering indices on the Riemann tensor, as well as using its symmetries. For example, $R_{\theta\theta}$ is:

$$R_{\theta\theta} = R^{\rho}{}_{\theta\rho\theta} = R^{\phi}{}_{\theta\phi\theta} = -R_{\theta}{}^{\phi}{}_{\phi\theta} = R_{\theta}{}^{\phi}{}_{\theta\phi} = g_{\theta\theta} g^{\phi\phi} R^{\theta}{}_{\phi\theta\phi} \ . \tag{2.5}$$

Then, using $g_{\theta\theta} = L^2$, $g^{\phi\phi} = L^{-2}\sin^{-2}\theta$, and $R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$, this becomes:

$$R_{\theta\theta} = 1 \ . \tag{2.6}$$

The other components are computed a little more straightforwardly:

$$R_{\phi\phi} = R^{\rho}{}_{\phi\rho\phi} = R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \ , \tag{2.7}$$

as well as

$$R_{\theta\phi} = R^{\rho}{}_{\theta\rho\phi} = R^{\theta}{}_{\theta\theta\phi} + R^{\phi}{}_{\theta\phi\phi} = 0 \ . \tag{2.8}$$

So, the Ricci tensor is diagonal!

Lastly, the Ricci scalar is obtained by fully contracting all indices on the Ricci tensor:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{1}{L^2}(1) + \frac{1}{L^2\sin^2\theta}(\sin^2\theta) = \frac{2}{L^2} \ . \tag{2.9}$$

So, the Ricci scalar is constant and inversely proportional to the size of the S^2. And, since the Riemann tensor, the Ricci tensor, and the Ricci scalar were all non-zero, we actually have a bona fide curved spacetime!

7.) The geodesic equations are presented compactly as

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \ . \tag{2.10}$$

Note that μ is a free index, so this actually represents multiple equations. In the case at hand of S^2, there is one for $\mu = \theta$ and $\mu = \phi$. The equations are simple plug-and-chug. For $\mu = \theta$, we find

$$\begin{aligned} 0 &= \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} \\ &= \frac{d^2\theta}{d\lambda^2} + \Gamma^\theta_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= \ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 \ , \end{aligned} \tag{2.11}$$

where we have introduce short-hand wherein a single dot means one derivative with respect to λ, and two dots means two derivatives with respect to λ. For $\mu = \phi$, the geodesic equation is

$$\begin{aligned} 0 &= \frac{d^2\phi}{d\lambda^2} + \Gamma^\phi_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} \\ &= \frac{d^2\phi}{d\lambda^2} + \Gamma^\phi_{\phi\theta}\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} + \Gamma^\phi_{\theta\phi}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} \\ &= \frac{d^2\phi}{d\lambda^2} + 2\Gamma^\phi_{\phi\theta}\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} \\ &= \ddot{\phi} + 2\cot\theta\,\dot{\phi}\,\dot{\theta} \ . \end{aligned} \tag{2.12}$$

Now, consider setting ϕ to a constant, i.e. indepndent of λ, such that $\dot{\phi} = 0$. Then, the $\mu = \phi$ geodesic equation is trivially satisfied, while the $\mu = \theta$ equation becomes $\ddot{\theta} = 0$. The solution is

$$\theta(\lambda) = c_1\lambda + c_2 \ , \tag{2.13}$$

for some constants c_1 and c_2. The resulting curves are simply great circles that go continuously through the north and south poles, as we keep increasing λ.

When $\theta = \pi/2$, then $\cos\theta = \cot\theta = 0$, while $\sin\theta = 1$. Additionally, θ is constant, so $\ddot{\theta} = \dot{\theta} = 0$. So, the first equation becomes trivially satisfied, while the second becomes simply $\ddot{\phi} = 0$, which is solved by

$$\phi(\lambda) = c_3\lambda + c_4 \ . \tag{2.14}$$

This curve is a great circle that goes around the equator (i.e. $\theta = \pi/2$). It does not pass through the north and south poles, but it does hit antipodal points along the equator.

All the solutions so far are in fact great circles, albeit only along the equator or along a great circle that hits the north and south poles. But, there was nothing special about where we decided to place the north and south pole. We can easily choose a new set of coordinates on the sphere where the poles and the equator are rotated at some angle with respect to the original ones, and if we repeat the same process we'll find geodesic great circles that go through these new poles, or the new equator.

This argument of rotating the geodesics we found here in order to obtain all great circles isn't a fully rigorous proof, but it should be enough to convince you physically that all geodesics on S^2 are great circles.

3 Practice Makes Perfect

8.) For the given spacetime, we have a diagonal metric $g_{tt} = -f(r)$ and $g_{rr} = h(r)$, and so $g^{tt} = -1/f(r)$ and $g^{rr} = 1/h(r)$.

Since all metric components depend only on r, all non-zero Christoffel symbols will have to have at least one occurence of r. Additionally, since all metric components are independent of t, t must appear either zero times or twice in the non-zero Christoffel symbols. If it only appears once, then we get terms like $\partial_t g_{rr}$ in the symbol, which is obviously zero.

The three independent, non-zero Christoffel symbols are thus

$$\Gamma^r_{tt} \ , \quad \Gamma^r_{rr} \ , \quad \Gamma^t_{tr} \ , \tag{3.1}$$

with $\Gamma^t_{rt} = \Gamma^t_{tr}$ obtained from these by symmetry. We can compute these three independent symbols fairly easily:

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{1}{2} g^{rr} \left(-\partial_r g_{tt}\right) = \frac{f'}{2h} \ , \\
\Gamma^r_{rr} &= \frac{1}{2} g^{rr} \left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr}\right) = \frac{1}{2} g^{rr} \left(\partial_r g_{rr}\right) = \frac{h'}{2h} \ , \\
\Gamma^t_{tr} &= \frac{1}{2} g^{tt} \left(\partial_r g_{tt}\right) = -\frac{f'}{2f} \ .
\end{aligned} \tag{3.2}$$

For the Christoffel symbol, you should be able to convince yourself that the only independent component is ${R^r}_{trt}$. Any component where r or t appear more than twice will vanish by antisymmetry. Explicitly, this is computed as follows:

$$\begin{aligned}
{R^r}_{trt} &= \partial_r \Gamma^r_{tt} - \partial_t \Gamma^r_{tr} + \Gamma^r_{r\lambda}\Gamma^\lambda_{tt} - \Gamma^r_{t\lambda}\Gamma^\lambda_{tr} \\
&= \partial_r \left(\frac{f'}{2h}\right) - \partial_t \left(0\right) + \Gamma^r_{rr}\Gamma^r_{tt} - \Gamma^r_{tt}\Gamma^t_{tr} \\
&= \frac{f''}{2h} - \frac{f'h'}{2h^2} + \left(\frac{h'}{2h}\right)\left(\frac{f'}{2h}\right) - \left(\frac{f'}{2h}\right)\left(-\frac{f'}{2f}\right) \\
&= \frac{f''}{2h} - \frac{f'h'}{4h^2} + \frac{f'^2}{4fh} \ .
\end{aligned} \tag{3.3}$$

The components of the Ricci tensor are:

$$\begin{aligned}
R_{tt} &= {R^\mu}_{t\mu t} = {R^r}_{trt} = \frac{f''}{2h} - \frac{f'h'}{4h^2} + \frac{f'^2}{4fh} \ , \\
R_{rr} &= {R^\mu}_{r\mu r} = {R^t}_{rtr} = {R_r}^t{}_{rt} = g_{rr} g^{tt} {R^r}_{trt} \\
&= h \times -\frac{1}{f} \times \left(\frac{f''}{2h} - \frac{f'h'}{4h^2} + \frac{f'^2}{4fh}\right) = -\frac{f''}{2f} + \frac{f'h'}{4fh} - \frac{f'^2}{4f^2} \ , \\
R_{tr} &= {R^\mu}_{t\mu r} = {R^t}_{ttr} + {R^r}_{trr} = g^{tt} R_{tttr} + g^{rr} R_{rtrr} = 0 \ .
\end{aligned} \tag{3.4}$$

Finally, the Ricci scalar is:

$$\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + g^{rr} R_{rr} \\
&= -\frac{1}{f}\left(\frac{f''}{2h} - \frac{f'h'}{4h^2} + \frac{f'^2}{4fh}\right) + \frac{1}{h}\left(-\frac{f''}{2f} + \frac{f'h'}{4fh} - \frac{f'^2}{4f^2}\right) \\
&= -\frac{f''}{fh} + \frac{f'h'}{2fh^2} - \frac{f'^2}{2f^2h} \ .
\end{aligned} \tag{3.5}$$

Like I said, not pretty. But, if you got this far, you should know how awesome this is: you have computed the curvature of a huge class of spacetimes, for arbitrary functions f and

h. Just by knowing the functional form of the metric, that's enough to derive explicitly how the curvature depends on derivatives of the functions f and h that show up in the metric.

Lastly, the two geodesic equations (one for t, one for r) are as follows:

$$\begin{aligned}
0 &= \ddot{t} + 2\Gamma^t_{tr}\dot{t}\dot{r} \\
&= \ddot{t} - \frac{f'}{f}\dot{t}\dot{r} \ , \\
0 &= \ddot{r} + \Gamma^r_{rr}\dot{r}^2 + \Gamma^r_{tt}\dot{t}^2 \\
&= \ddot{r} + \frac{h'}{2h}\dot{r}^2 + \frac{f'}{2h}\dot{t}^2 \ .
\end{aligned} \tag{3.6}$$

9.) Since $x = f(x')$, the chain rule immediately tells us that

$$dx = \frac{\partial x}{\partial x'}dx' = f'(x')dx' \ , \tag{3.7}$$

and so the metric is

$$ds^2 = dx^2 = f'(x')^2 dx'^2 \ . \tag{3.8}$$

That is, $g_{x'x'} = f'(x')^2$, and correspondingly $g^{x'x'} = \frac{1}{f'(x')^2}$. Since the only coordinate is x' in this new frame, the only Christoffel symbol is

$$\begin{aligned}
\Gamma^{x'}_{x'x'} &= \frac{1}{2}g^{x'x'}\left(\partial_{x'}g_{x'x'} + \partial_{x'}g_{x'x'} - \partial_{x'}g_{x'x'}\right) \\
&= \frac{1}{2}g^{x'x'}\left(\partial_{x'}g_{x'x'}\right) \\
&= \frac{1}{2f'(x')^2} \cdot \partial_{x'}\left(f'(x')^2\right) \\
&= \frac{1}{2f'(x')^2} \cdot 2f'(x')f''(x') \\
&= \frac{f''(x')}{f'(x')} \ .
\end{aligned} \tag{3.9}$$

Since this is the only Christoffel symbol, the geodesic equation is very simple:

$$0 = \frac{d^2x'}{d\lambda^2} + \Gamma^{x'}_{x'x'}\left(\frac{dx'}{d\lambda}\right)^2 = \frac{d^2x'}{d\lambda^2} + \frac{f''(x')}{f'(x')}\left(\frac{dx'}{d\lambda}\right)^2 \ . \tag{3.10}$$

This is what we set out to prove.

Now, what about the Riemann tensor? Since the only coordinate is x', the only component of the Riemann tensor is $R_{x'x'x'x'}$, which must be zero in order to retain its antisymmetry in the first and second pairs of indices. So, it seems that the Riemann tensor is trivially zero.

This is not specific to our setup, either. In any 1D space, there is no Riemann tensor simply because there are not enough indices to be compatible with the symmetries of the Riemann tensor. And, since we define the Ricci tensor and Ricci scalar via contractions of the Riemann tensor, these quantities are trivially zero as well.